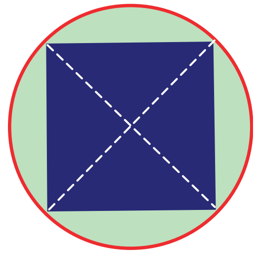
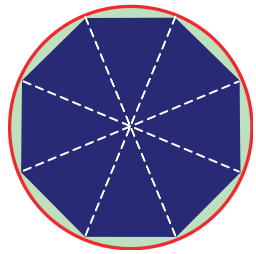
1.3

Limits



The Greek mathematician Archimedes (c. 287–212 B.C.) developed a proof of the formula for the area of a circle, $A = \pi r^2$. His method, known as the "method of exhaustion," involved calculating the area of regular polygons (meaning their sides are equal) that were *inscribed* in the circle. This means that they were drawn inside the circle such that their vertices touched the circumference, as shown in the diagram. The area of the polygon provided an estimate of the area of the circle. As Archimedes increased the number of sides of the polygon, its shape came closer to the shape of a circle. For example, as shown here, an octagon provides a much better estimate of the area of a circle than a square does. The area of a hexadecagon, a polygon with 16 sides, would provide an even better estimate, and so on. What about a myriagon, a polygon with 10 000 sides? What happens to the estimate as the number of sides approaches infinity?



Archimedes' method of finding the area of a circle is based on the concept of a **limit**. The circle is the limiting shape of the polygon. As the number of sides gets larger, the area of the polygon approaches its limit, the shape of a circle, without ever becoming an actual circle. In Section 1.2, you used a similar strategy to estimate the instantaneous rate of change of a function at a single point. Your estimate became increasingly accurate as the interval between two points was made smaller. Using limits, the interval can be made infinitely small, approaching zero. As this happens, the slope of the secant approaches its limiting value—the slope of the tangent. In this section, you will explore limits and methods for calculating them.

Investigate A How can you determine the limit of a sequence?

Tools
- graphing calculator

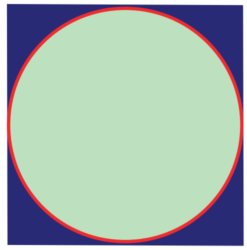
Optional
- *Fathom*™

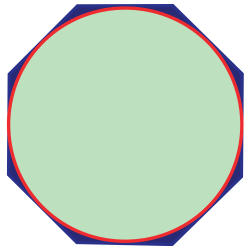
CONNECTIONS

An infinite sequence sometimes has a limiting value, L. This means that as n gets larger, the terms of the sequence, t_n, get closer to L. Another way of saying "as n gets larger" is, "as n approaches infinity." This can be written $n \to \infty$. The symbol ∞ does not represent a particular number, but it may be helpful to think of ∞ as a very large positive number.

1. Examine the terms of the infinite sequence $1, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10\,000}, \ldots$. The general term of this sequence is $t_n = \frac{1}{10^n}$. What happens to the value of each term as n increases and the denominator becomes larger?
2. What is the value of $\lim_{n\to\infty} t_n$ (read "the limit of t_n as n approaches infinity.")? Why can you then say that its limit exists?
3. Plot ordered pairs, $\left(n, \frac{1}{10^n}\right)$, that correspond to the sequence. Describe how the graph confirms your answer in step 2.
4. **Reflect** Explain why $\lim_{n\to\infty} t_n$ expresses a value that is approached, but not reached.
5. **Reflect** Examine the terms of the infinite sequence 1, 4, 9, 16, 25, 36, … , n^2, … . Explain why $\lim_{n\to\infty} t_n$ does not exist for this sequence.

In the development of the formula $A = \pi r^2$, Archimedes not only approached the area of the circle from the inside, but from the outside as well. He calculated the area of a regular polygon that *circumscribed* the circle, meaning that it surrounded the circle, with each of its sides touching the circle's circumference. As the number of sides to the polygon was increased, its shape and area became closer to that of a circle.





Archimedes' approach can also be applied to determining the limit of a function. A limit can be approached from the left side and from the right side, called **left-hand limits** and **right-hand limits**. To evaluate a left-hand limit, we use values that are smaller than, or on the left side of the value being approached. To evaluate a right-hand limit we use values that are larger than, or on the right side of the value being approached. In either case, the value is very close to the approached value.

Investigate B How can you determine the limit of a function from its equation?

Tools

- graphing calculator

Optional

- *Fathom*™
- computer with *The Geometer's Sketchpad*®

1. a) Copy and complete the table for the function $y = x^2 - 2$

x	2	2.5	2.9	2.99	2.999	3	3.001	3.01	3.1	3.5	4
$y = x^2 - 2$											

b) Examine the values in the table that are to the left of 3, beginning with $x = 2$. What value is y approaching as x gets closer to, or approaches, 3 from the left?

c) Beginning at $x = 4$, what value is y approaching as x approaches 3 from the right?

d) **Reflect** Compare the values you determined for y in parts b) and c). What do you notice?

2. a) Graph $y = x^2 - 2$.

b) Press the TRACE key and trace along the curve toward $x = 3$ from the left. What value does y approach as x approaches 3?

c) Use TRACE to trace along the curve toward $x = 3$ from the right. What value does y approach as x approaches 3?

d) **Reflect** How does the graph support your results in question 1?

3. **Reflect** The value that y approaches as x approaches 3 is "the limit of the function $y = x^2 - 2$ as x approaches 3," written as $\lim_{x \to 3}(x^2 - 2)$. Does it make sense to say, "the limit of $y = x^2 - 2$ exists at $x = 3$"?

CONNECTIONS

To see an animation of step 2 of this Investigate, go to *www.mcgrawhill.ca/links/calculus12*, and follow the links to Section 1.3.

It was stated earlier that the limit exists if the sequence approaches a single value. More accurately, the limit of a function exists at a point if both the right-hand and left-hand limits exist and they both approach the *same* value.

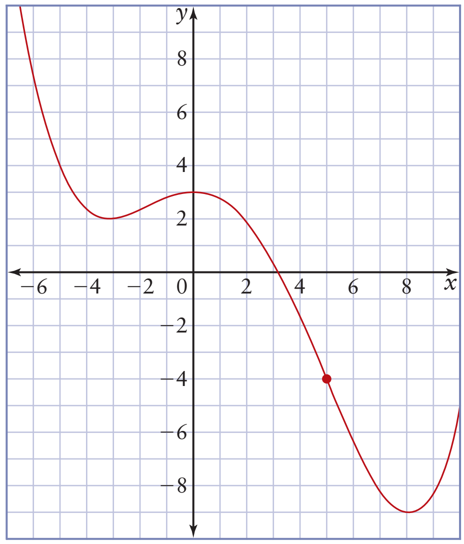
$\lim_{x \to a} f(x)$ exists if the following three criteria are met:

1. $\lim_{x \to a^-} f(x)$ exists
2. $\lim_{x \to a^+} f(x)$ exists
3. $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

Investigate C How can you determine the limit of a function from a given graph?

CONNECTIONS

To see an animated example of two-sided limits, go to *www.mcgrawhill.ca/links/calculus12*, and follow the links to Section 1.3.



1. a) Place your fingertip on the graph at $x = -6$ and trace the graph approaching $x = 5$ from the left. State the y-value that is being approached. This is the left-hand limit.

   b) Place your fingertip on the graph corresponding to $x = 9$, and trace the graph approaching $x = 5$ from the right. State the y-value that is being approached. This is the right-hand limit.

   c) **Reflect** What does the value $f(5)$ represent for this curve?

2. **Reflect** Trace the entire curve with your finger. Why would it make sense to refer to a curve like this as continuous? Explain why all polynomial functions would be continuous.

3. **Reflect** Explain the definition of *continuous* provided in the box below.

A function $f(x)$ is **continuous at a number** $x = a$ if the following three conditions are satisfied:

1. $f(a)$ is defined
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

A **continuous function** is a function that is continuous at x, for all values of $x \in \mathbb{R}$. Informally, a function is continuous if you can draw its graph without lifting your pencil. If the curve has holes or gaps, it is **discontinuous**, or has a **discontinuity**, at the point at which the gap occurs. You cannot draw this function without lifting your pencil. You will explore discontinuous functions in Section 1.4.

Example 1 Apply Limits to Analyse the End Behaviour of a Sequence

For each of the following sequences,

i) state the limit, if it exists. If it does not exist, explain why. Use a graph to support your answer.

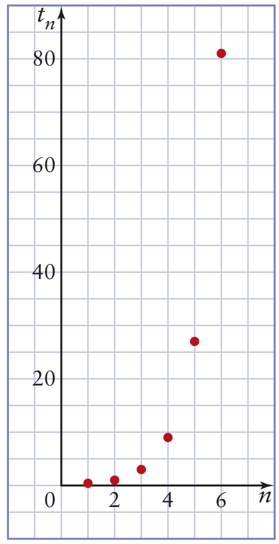
ii) write a limit expression to represent the end behaviour of the sequence.

a) $\frac{1}{3}, 1, 3, 9, 27, \ldots, 3^{n-2}, \ldots$

b) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots, \frac{n}{n+1}, \ldots$

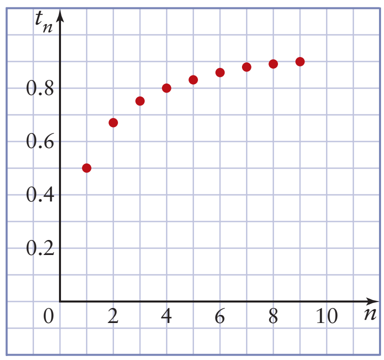
Solution

a) i) Examine the terms of the sequence $\frac{1}{3}, 1, 3, 9, 27, \ldots, 3^{n-2}, \ldots$. Since the terms are increasing and not converging to a value, the sequence does not have a limit. This fact is verified by the graph obtained by plotting the points $(n, t_n): \left(1, \frac{1}{3}\right), (2, 1), (3, 3), (4, 9), (5, 27), (6, 81), \ldots$.



ii) The end behaviour of the sequence is represented by the limit expression $\lim_{n\to\infty} 3^{n-2} = \infty$. The infinity symbol indicates that the terms of the sequence are becoming larger positive values, or increasing without bound, and so the limit does not exist.

b) i) Examine the terms of the sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots, \frac{n}{n+1}, \ldots$.
Convert each term to a decimal, the sequence becomes 0.5, 0.67, 0.75, 0.8, 0.83, … . The next three terms of the sequence are $\frac{6}{7}$, $\frac{7}{8}$, and $\frac{8}{9}$, or 0.857, 0.875, and 0.889. Though the terms are increasing, they are not increasing without bound—they appear to be approaching 1 as n becomes larger.



This can be verified by determining $t_n = \frac{n}{n+1}$ for large values of n. $t_{100} = \frac{100}{101} = 0.99$ and $t_{1000} = \frac{1000}{1001} = 0.999$. The value of this function will never become larger than 1 because the value of the numerator for this function is always one less than the value of the denominator.

ii) The end behaviour of the terms of the sequence is represented by the limit expression $\lim_{n\to\infty} \frac{n}{n+1} = 1$.

Example 2 Analyse a Graph to Evaluate the Limit of a Function

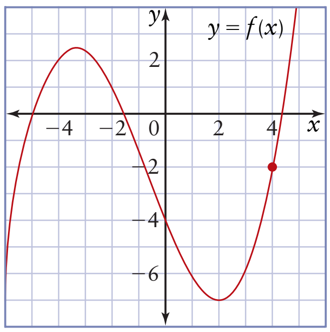
Determine the following for the graph below.

a) $\lim_{x\to 4^-} f(x)$

b) $\lim_{x\to 4^+} f(x)$

c) $\lim_{x\to 4} f(x)$

d) $f(4)$



Solution

a) $\lim_{x\to 4^-} f(x)$ refers to the limit as x approaches 4 from the left. Tracing along the graph from the left, you will see that the y-value that is being approached at $x = 4$ is -2.

b) $\lim_{x\to 4^+} f(x)$ refers to the limit as x approaches 4 from the right. Tracing along the graph from the right, you will see that the y-value that is being approached at $x = 4$ is -2.

c) Both the left-hand and right-hand limits equal –2, thus, $\lim_{x\to 4} f(x) = -2$. Also, the conditions for continuity are satisfied, so $f(x)$ is continuous at $x = -2$.

d) $f(4) = -2$.

KEY CONCEPTS

- A sequence is a function, $f(n) = t_n$, whose domain is the set of natural numbers $\mathbb{N}$.
- $\lim\limits_{x \to a} f(x)$ exists if the following three criteria are met:
  1. $\lim\limits_{x \to a^-} f(x)$ exists
  2. $\lim\limits_{x \to a^+} f(x)$ exists
  3. $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x)$
- $\lim\limits_{x \to a} f(x) = L$, which is read as "the limit of $f(x)$, as x approaches a, is equal to L."
- If $\lim\limits_{x \to a^-} f(x) \neq \lim\limits_{x \to a^+} f(x)$, then $\lim\limits_{x \to a} f(x)$ does not exist.
- A function $f(x)$ is continuous at a value $x = a$ if the following three conditions are satisfied:
  1. $f(a)$ is defined, that is, a is in the domain of $f(x)$
  2. $\lim\limits_{x \to a} f(x)$ exists
  3. $\lim\limits_{x \to a} f(x) = f(a)$

Communicate Your Understanding

C1 Describe circumstances when the limit of a sequence exists, and when it does not exist.

C2 What information do the left-hand limit and right-hand limit provide about the graph of a function?

C3 How can you tell if a function is continuous or discontinuous from its graph?

C4 How do limits help determine if a function is continuous?

A Practise

1. State the limit of each sequence, if it exists. If it does not exist, explain why.

a) 1, −1, 1, −1, 1, −1, 1, −1, …

b) 5.9, 5.99, 5.999, 5.999 9, 5.999 99, 5.999 999, …

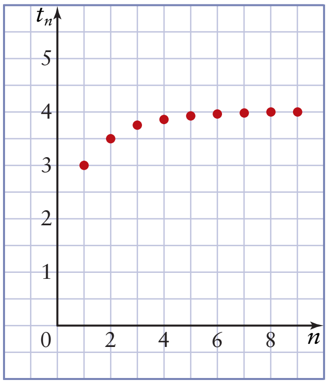
c) $-2, 0, -1, 0, -\frac{1}{2}, 0, -\frac{1}{4}, 0, -\frac{1}{8}, 0, -\frac{1}{16}, 0, \cdots$

d) 3.1, 3.01, 3.001, 3.000 1, 3.000 01, …

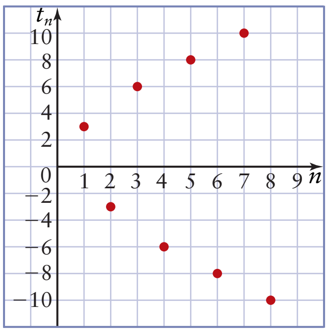
e) −3, −2.9, −3, −2.99, −3, −2.999, −3, −2.999 9, …

2. State the limit of the sequence represented by each graph, if it exists. If it does not exist, explain why.

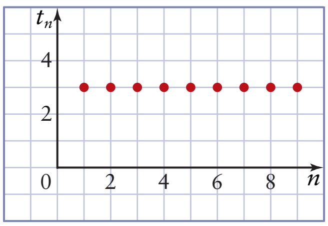
a)



b)



c)



3. Given that $\lim_{x\to 1^-} f(x) = 4$ and $\lim_{x\to 1^+} f(x) = -4$, what is true about $\lim_{x\to 1} f(x)$?

4. Given that $\lim_{x\to -3^-} f(x) = 1$, $\lim_{x\to -3^+} f(x) = 1$, and $f(-3) = 1$, what is true about $\lim_{x\to -3} f(x)$?

5. Given that $\lim_{x\to 2^-} f(x) = 0$, $\lim_{x\to 2^+} f(x) = 0$, and $f(2) = 3$, what is true about $\lim_{x\to 2} f(x)$?

6. Consider a function $y = f(x)$ such that $\lim_{x\to 3^-} f(x) = 2$, $\lim_{x\to 3^+} f(x) = 2$, and $f(3) = -1$. Explain whether each statement is true or false.

   a) $y = f(x)$ is continuous at $x = 3$.

   b) The limit of $f(x)$ as x approaches 3 does not exist.

   c) The value of the left-hand limit is 2.

   d) The value of the right-hand limit is -1.

   e) When $x = 3$, the y-value of the function is 2.

7. **a)** What is true about the graph of $y = h(x)$, given that $\lim_{x\to -1^-} h(x) = \lim_{x\to -1^+} h(x) = 1$, and $h(-1) = 1$?

   b) What is true if $\lim_{x\to -1^-} h(x) = 1$, $\lim_{x\to -1^+} = -1$, and $h(-1) = 1$?

B Connect and Apply

8. The general term of a particular infinite sequence is $\frac{2}{3^n}$.

   a) Write the first six terms of the sequence.

   b) Explain why the limit of the sequence is 0.

9. The general term of a particular infinite sequence is $n^3 - n^2$.

   a) Write the first six terms of the sequence.

   b) Explain why the limit of the sequence does not exist.

10. What special number is represented by the limit of the sequence 3, 3.1, 3.14, 3.141, 3.141 5, 3.141 59, 3.141 592, … ?

11. What fraction is equivalent to the limit of the sequence 0.3, 0.33, 0.333, 0.333 3, 0.333 33, …?

12. For each of the following sequences,

   i) State the limit, if it exists. If it does not exist, explain why. Use a graph to support your answer.

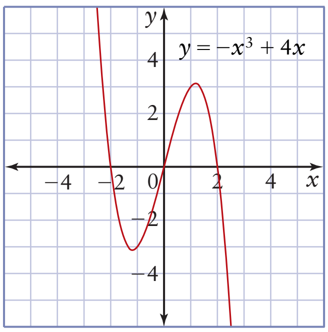
ii) Write a limit expression to represent the behaviour of the sequence.

a) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots, \frac{1}{n}, \ldots$

b) $2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, 2^{2-n}, \ldots$

c) $4, 5\frac{1}{2}, 4\frac{2}{3}, 5\frac{1}{4}, 4\frac{4}{5}, 5\frac{1}{6}, \ldots, 5+\frac{(-1)^n}{n}, \ldots$

13. Examine the given graph.



a) State the domain of the function.

b) Evaluate the following

i) $\lim_{x \to -2^-} (-x^3 + 4x)$ **ii)** $\lim_{x \to -2^+} (-x^3 + 4x)$

iii) $\lim_{x \to -2} (-x^3 + 4x)$ **iv)** $f(-2)$

c) Is the graph continuous at $x = -2$? Justify your answer.

14. The period of a pendulum is approximately represented by the function $T(l) = 2\sqrt{l}$, where T is time, in seconds, and l is the length of the pendulum, in metres.

a) Evaluate $\lim_{l \to -0^+} 2\sqrt{l}$.

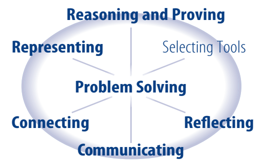
b) Interpret the meaning of your result in part a).

c) Graph the function. How does the graph support your result in part a)?

CONNECTIONS

The ratios of consecutive terms of the Fibonacci sequence approach the golden ratio, $\frac{\sqrt{5}+1}{2}$, also called the golden mean or golden number. How is this number related to your results in question 15? Research this "heavenly number" to find out more about it, and where it appears in nature, art, and design.

15. A **recursive sequence** is a sequence where the nth term, t_n, is defined in terms of preceding terms, t_{n-1}, t_{n-2}, etc.



One of the most famous recursive sequences is the Fibonacci sequence, created by Leonardo Pisano (1170–1250). The terms of this sequence are defined as follows: $f_1 = 1$, $f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$, where $n \geq 3$.

a) Copy and complete the table. In the fourth column evaluate each ratio to six decimal places.

n	f_n	$\frac{f_n}{f_{n-1}}$	Decimal
1	1		
2	1	$\frac{1}{1}$	
3	2	$\frac{2}{1}$	
4	3		
5			
6			
7			
8			
9			
10			

b) What value do the ratios approach? Predict the value of the limit of the ratios.

c) Calculate three more ratios.

d) Graph the ordered pairs (1, 1), (2, 2), … , (n, r_n), where n represents the ratio number and r_n is the ratio value, to six decimal places. How does your graph confirm the value in part b)?

e) Write an expression, using a limit, to represent the value of the ratios of consecutive terms of the Fibonacci sequence.

16. a) Construct a table of values to determine each limit in parts i) and ii), and then use your results to determine the limit in part iii).

i) $\lim\limits_{x \to 4^-} \sqrt{4 - x}$

ii) $\lim\limits_{x \to 4^+} \sqrt{4 - x}$

iii) $\lim\limits_{x \to 4} \sqrt{4 - x}$

b) **Use Technology** Graph the function from part a) using a graphing calculator. How does the graph support your results in part a)?

17. a) Construct a table of values to determine each limit in parts i) and ii), and then use your results to determine the limit in part iii).

i) $\lim\limits_{x \to -2^-} \sqrt{x + 2}$

ii) $\lim\limits_{x \to -2^+} \sqrt{x + 2}$

iii) $\lim\limits_{x \to -2} \sqrt{x + 2}$

b) **Use Technology** Graph the function from part a) using a graphing calculator. How does the graph support your results in part a)?

C Extend and Challenge

18. a) Suppose \$1 is deposited for 1 year into an account that pays an interest rate of 100%. What is the value of the account at the end of 1 year for each compounding period?

i) annual **ii)** semi-annual

iii) monthly **iv)** daily

v) every minute **vi)** every second

b) How is the above situation related to the following sequence?

$$\left(1 + \frac{1}{1}\right)^1, \left(1 + \frac{1}{2}\right)^2, \left(1 + \frac{1}{3}\right)^3, \left(1 + \frac{1}{4}\right)^4,$$
$$\ldots, \left(1 + \frac{1}{n}\right)^n, \ldots$$

c) Do some research to find out how the limit of the above sequence relates to Euler's number, e.

19. A **continued fraction** is an expression of the form

$$x = a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{a_4 + \cfrac{b_4}{a_5 + \cfrac{b_5}{a_6 + \cfrac{b_6}{\ddots}}}}}}$$

where a_1 is an integer, and all the other numbers a_n and b_n are positive integers. Determine the limit of the following continued fraction. What special number does the limit represent?

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}}}$$

20. Determine the limit of the following sequence. (Hint: Express each term as a power of 3.)

$$\sqrt{3}, \sqrt{3\sqrt{3}}, \sqrt{3\sqrt{3\sqrt{3}}}, \sqrt{3\sqrt{3\sqrt{3\sqrt{3}}}}, \sqrt{3\sqrt{3\sqrt{3\sqrt{3\sqrt{3}}}}}, \ldots$$

21. **Math Contest** The sequence *cat, nut, not, act, art, bat,* … is a strange arithmetic sequence in which each letter represents a unique digit. Determine the next "word" in the sequence.

22. **Math Contest** Determine the value(s) of k if $(3, k)$ is a point on the curve $x^2y - y^2x = 30$.