7.3 Applications of the Dot Product

The dot product has many applications in mathematics and science. Finding the work done, determining the angle between two vectors, and finding the projection of one vector onto another are just three of these.

Example 1 Find the Work Done

Angela has entered the wheelchair division of a marathon race. While training, she races her wheelchair up a 300-m hill with a constant force of 500 N applied at an angle of 30° to the surface of the hill. Find the work done by Angela, to the nearest 100 J.

Solution

Let $\vec{f}$ represent the force vector and $\vec{s}$ represent the displacement vector. The work done, W, is given by

$$\begin{aligned} W &= \vec{f} \cdot \vec{s} \\ &= |\vec{f}||\vec{s}|\cos\theta \\ &= (500)(300)\cos 30° \\ &\doteq 129\ 904 \end{aligned}$$

The work done by Angela is approximately 129 900 N·m or 129 900 J.

Example 2 Find the Angle Between Two Vectors

Determine the angle between the vectors in each pair.

a) $\vec{g} = [5, 1]$ and $\vec{h} = [-3, 8]$ **b)** $\vec{s} = [-3, 6]$ and $\vec{t} = [4, 2]$

Solution

a) From the definition of the dot product,

$$\vec{g} \cdot \vec{h} = |\vec{g}||\vec{h}|\cos\theta$$

$$\cos\theta = \frac{\vec{g} \cdot \vec{h}}{|\vec{g}||\vec{h}|}$$

$$\cos\theta = \frac{5(-3) + 1(8)}{\sqrt{5^2 + 1^2}\sqrt{(-3)^2 + 8^2}}$$ Use Cartesian vectors.

$$\theta = \cos^{-1}\left(\frac{-7}{\sqrt{26}\sqrt{73}}\right)$$

$$\theta \doteq 99.246°$$

The angle between $\vec{g} = [5, 1]$ and $\vec{h} = [-3, 8]$ is approximately 99.2°.

b) $\cos\theta = \dfrac{\vec{s}\cdot\vec{t}}{|\vec{s}||\vec{t}|}$

$\cos\theta = \dfrac{-3(4)+6(2)}{\sqrt{(-3)^2+6^2}\sqrt{4^2+2^2}}$

$\cos\theta = \dfrac{0}{\sqrt{45}\sqrt{20}}$

$\cos\theta = 0$

$\theta = 90°$

The angle between $\vec{s} = [-3, 6]$ and $\vec{t} = [4, 2]$ is 90°. These vectors are orthogonal.

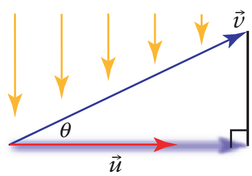
To find the angle, θ, between two Cartesian vectors $\vec{u}$ and $\vec{v}$, use the formula $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$.

Vector Projections

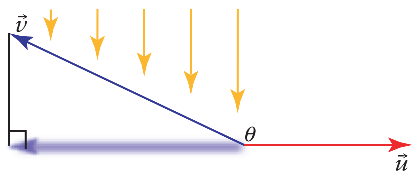
You can think of a vector projection like a shadow. Consider the following diagram, where the vertical arrows represent light from above.

Think of the **projection of $\vec{v}$ on $\vec{u}$** as the shadow that $\vec{v}$ casts on $\vec{u}$.

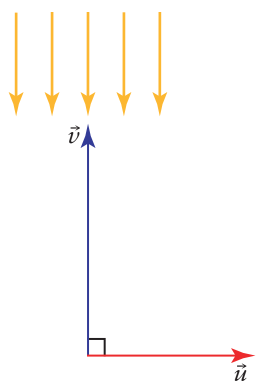
If the angle between $\vec{v}$ and $\vec{u}$ is less than 90°, then the projection of $\vec{v}$ on $\vec{u}$, or $\text{proj}_{\vec{u}}\,\vec{v}$, is the vector component of $\vec{v}$ in the direction of $\vec{u}$.



If the angle between $\vec{v}$ and $\vec{u}$ is between 90° and 180°, the direction of $\text{proj}_{\vec{u}}\,\vec{v}$ is opposite to the direction of $\vec{u}$.



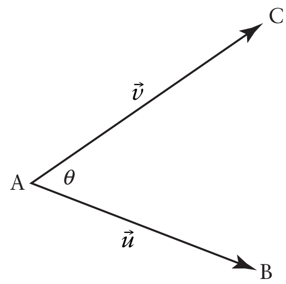
If $\vec{v}$ is perpendicular to $\vec{u}$, then $\vec{v}$ casts no "shadow." So, if $\theta = 90°$, $\text{proj}_{\vec{u}}\,\vec{v} = \vec{0}$.



Example 3 Find the Projection of One Vector on Another

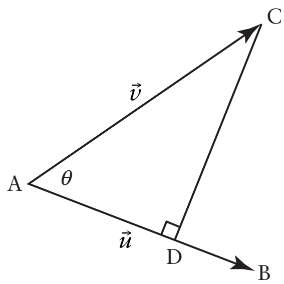
a) Find the projection of $\vec{v}$ on $\vec{u}$ if $0 < \theta < 90°$.

b) Find $\text{proj}_{\vec{u}}\vec{v}$ if $90° < \theta < 180°$.



Solution

a) Construct segment CD perpendicular to $\overrightarrow{AB}$, with D on $\overrightarrow{AB}$.



Then, $\overrightarrow{AD} = \text{proj}_{\vec{u}}\vec{v}$. From the triangle,

$$\frac{|\overrightarrow{AD}|}{|\vec{v}|} = \cos\theta$$

$$|\overrightarrow{AD}| = |\vec{v}|\cos\theta$$

$$|\text{proj}_{\vec{u}}\vec{v}| = |\vec{v}|\cos\theta$$

CONNECTIONS

In question 13, you will show that an equivalent formula for the magnitude of the projection of $\vec{v}$ on $\vec{u}$ is $|\text{proj}_{\vec{u}}\vec{v}| = \frac{\vec{v}\cdot\vec{u}}{|\vec{u}|}$.

This gives the magnitude of the projection, which is also called the scalar component of $\vec{v}$ on $\vec{u}$. The direction of $\text{proj}_{\vec{u}}\vec{v}$ is the same as the direction of $\vec{u}$. If we multiply the magnitude of $\text{proj}_{\vec{u}}\vec{v}$ by a unit vector in the direction of $\vec{u}$, we will get the complete form of $\text{proj}_{\vec{u}}\vec{v}$. Let $k\vec{u}$ be the unit vector in the direction of $\vec{u}$. Then,

$$|k\vec{u}| = 1$$

$$|k||\vec{u}| = 1$$

$$|k| = \frac{1}{|\vec{u}|}$$

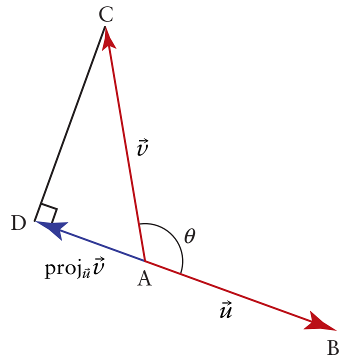
Since we want the unit vector to be in the same direction as $\vec{u}$, k must be positive. Thus, $k = \frac{1}{|\vec{u}|}$.

So, a unit vector in the direction of $\vec{u}$ is $\frac{1}{|\vec{u}|}\vec{u}$. Thus,

$$\text{proj}_{\vec{u}}\vec{v} = \underbrace{|\vec{v}|\cos\theta}_{\text{magnitude}}\underbrace{\left(\frac{1}{|\vec{u}|}\vec{u}\right)}_{\text{direction}}$$

b) If $90° < \theta < 180°$, then, from the diagram,

$$\begin{aligned}|\text{proj}_{\vec{u}}\,\vec{v}| &= |\vec{v}|\cos(180° - \theta)\\ &= |\vec{v}|\,(-\cos\theta)\\ &= -|\vec{v}|\cos\theta\end{aligned}$$



The direction of $\text{proj}_{\vec{u}}\,\vec{v}$ is opposite to the direction of $\vec{u}$.

Thus, we want the negative value of k from part a).

$$k = -\frac{1}{|\vec{u}|}$$

So, a unit vector in the opposite direction to $\vec{u}$ is $-\frac{1}{|\vec{u}|}\vec{u}$. Thus,

$$\text{proj}_{\vec{u}}\,\vec{v} = \underbrace{-|\vec{v}|\cos\theta}_{\text{magnitude}}\underbrace{\left(-\frac{1}{|\vec{u}|}\vec{u}\right)}_{\text{direction}}$$

$$= |\vec{v}|\cos\theta\left(\frac{1}{|\vec{u}|}\vec{u}\right)$$

This is the same formula as in part a).

CONNECTIONS

In question 14, you will show that an equivalent formula for the projection of $\vec{v}$ on $\vec{u}$ is $\text{proj}_{\vec{u}}\,\vec{v} = \left(\frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\right)\vec{u}$.

The projection of $\vec{v}$ on $\vec{u}$, where θ is the angle between $\vec{v}$ and $\vec{u}$, is

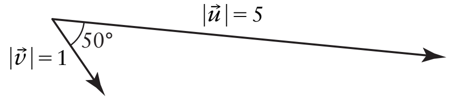
$\text{proj}_{\vec{u}}\,\vec{v} = |\vec{v}|\cos\theta\left(\frac{1}{|\vec{u}|}\vec{u}\right)$ or $\text{proj}_{\vec{u}}\,\vec{v} = \left(\frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\right)\vec{u}$.

If $0 < \theta < 90°$, $|\text{proj}_{\vec{u}}\,\vec{v}| = |\vec{v}|\cos\theta$ or $|\text{proj}_{\vec{u}}\,\vec{v}| = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|}$.

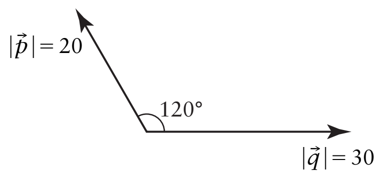
If $90° < \theta < 180°$, then $|\text{proj}_{\vec{u}}\,\vec{v}| = -|\vec{v}|\cos\theta$ or $|\text{proj}_{\vec{u}}\,\vec{v}| = -\frac{\vec{u}\cdot\vec{v}}{|\vec{u}|}$.

Example 4 Find the Projection of One Vector on Another

a) Determine the projection of $\vec{u}$ on $\vec{v}$.



b) Determine the projection of $\vec{p}$ on $\vec{q}$.



c) Determine the projection of $\vec{d} = [2, -3]$ on $\vec{c} = [1, 4]$.

d) Illustrate the projections in parts a) to d) geometrically.

Solution

a) Since $\theta < 90°$, the magnitude of the projection is given by

$$\begin{aligned} |\text{proj}_{\vec{v}}\, \vec{u}| &= |\vec{u}|\cos\theta \\ &= 5\cos 50° \\ &\doteq 3.21 \end{aligned}$$

The direction of $\text{proj}_{\vec{v}}\, \vec{u}$ is the same as the direction of $\vec{v}$.

b) Since $\theta > 90°$, the magnitude of the projection is given by

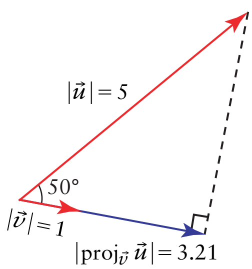
$$\begin{aligned} |\text{proj}_{\vec{q}}\, \vec{p}| &= -|\vec{p}|\cos\theta \\ &= -20\cos 120° \\ &= 10 \end{aligned}$$

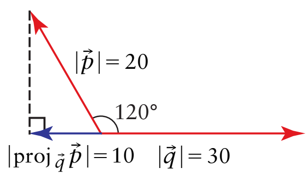
The direction of $\text{proj}_{\vec{q}}\, \vec{p}$ is opposite to the direction of $\vec{q}$.

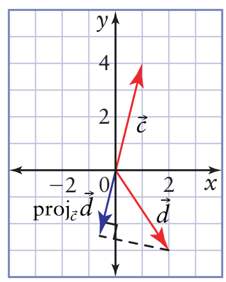
c)
$$\begin{aligned} \text{proj}_{\vec{c}}\, \vec{d} &= \left(\frac{\vec{d}\cdot\vec{c}}{\vec{c}\cdot\vec{c}}\right)\vec{c} \\ &= \left(\frac{[2, -3]\cdot[1, 4]}{[1, 4]\cdot[1, 4]}\right)[1, 4] \\ &= \left(\frac{2(1) + (-3)(4)}{1^2 + 4^2}\right)[1, 4] \\ &= \frac{-10}{17}[1, 4] \\ &= \left[-\frac{10}{17}, -\frac{40}{17}\right] \end{aligned}$$

Note that the direction of $\text{proj}_{\vec{c}}\, \vec{d}$ is opposite to the direction of $\vec{c}$.

d)







Example 5 Dot Product in Sales

A shoe store sold 350 pairs of Excalibur shoes and 275 pairs of Camelot shoes in a year. Excalibur shoes sell for \$175 and Camelot shoes sell for \$250.

a) Write a Cartesian vector, $\vec{s}$, to represent the numbers of pairs of shoes sold.

b) Write a Cartesian vector, $\vec{p}$, to represent the prices of the shoes.

c) Find the dot product $\vec{s}\cdot\vec{p}$. What does this dot product represent?

Solution

a) $\vec{s} = [350, 275]$

b) $\vec{p} = [175, 250]$

c)
$$\begin{aligned}\vec{s}\cdot\vec{p} &= [350, 275]\cdot[175, 250]\\ &= [350(175) + 275(250)]\\ &= 130\ 000\end{aligned}$$

The dot product represents the revenue, \$130 000, from sales of the shoes.

KEY CONCEPTS

- To find the angle, θ, between two Cartesian vectors $\vec{u}$ and $\vec{v}$, use the formula $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$.
- For two vectors $\vec{u}$ and $\vec{v}$ with an angle of θ between them, the projection of $\vec{v}$ on $\vec{u}$ is the vector component of $\vec{v}$ in the direction of $\vec{u}$:
  - $\text{proj}_{\vec{u}}\,\vec{v} = |\vec{v}|\cos\theta\left(\dfrac{1}{|\vec{u}|}\vec{u}\right)$ or $\text{proj}_{\vec{u}}\,\vec{v} = \left(\dfrac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\right)$
  - $|\text{proj}_{\vec{u}}\,\vec{v}| = |\vec{v}|\cos\theta$ if $0° < \theta < 90°$
  - $|\text{proj}_{\vec{u}}\,\vec{v}| = -|\vec{v}|\cos\theta$ if $90° < \theta < 180°$
  - $|\text{proj}_{\vec{u}}\,\vec{v}| = \dfrac{\vec{v}\cdot\vec{u}}{|\vec{u}|}$
- If $0 < \theta < 90°$, then $\text{proj}_{\vec{u}}\,\vec{v}$ is in the same direction as $\vec{u}$.
  If $90° < \theta < 180°$, then $\text{proj}_{\vec{u}}\,\vec{v}$ is in the opposite direction as $\vec{u}$.
  If $\theta = 90°$, then $\text{proj}_{\vec{u}}\,\vec{v}$ is the zero vector, $\vec{0}$.

Communicate Your Understanding

C1 There are two ways to calculate the magnitude of a projection. Describe when you would use each method.

C2 What happens to the projection of $\vec{v}$ on $\vec{u}$ if $\vec{v}$ is much longer than $\vec{u}$?

C3 What happens if the angle between $\vec{u}$ and $\vec{v}$ is close to, but less than, 180°? What if it is equal to 180°?

C4 Is it possible for the angle between two vectors to be more than 180°? How does the formula support this conclusion?

C5 At the gym, Paul lifts an 80-kg barbell a distance of 1 m above the floor and then lowers the barbell to the floor. How much mechanical work has he done?

A Practise

1. Determine the work done by each force, $\vec{F}$, in newtons, for an object moving along the vector, $\vec{s}$, in metres.

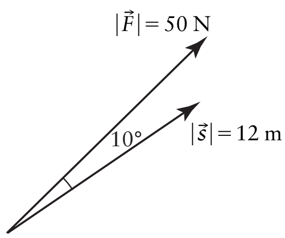
a) $\vec{F} = [5, 2]$, $\vec{s} = [7, 4]$

b) $\vec{F} = [100, 400]$, $\vec{s} = [12, 27]$

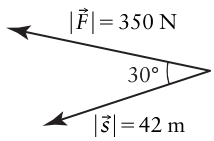
c) $\vec{F} = [67.8, 3.9]$, $\vec{s} = [4.7, 3.2]$

2. Determine the work done by the force, $\vec{F}$, in the direction of the displacement, $\vec{s}$.

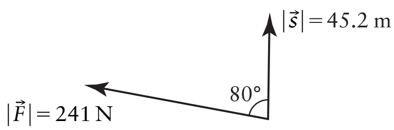
a)



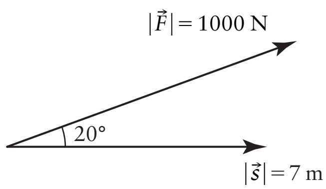
b)



c)



d)



3. Calculate the angle between the vectors in each pair. Illustrate geometrically.

a) $\vec{p} = [7, 8]$, $\vec{q} = [4, 3]$

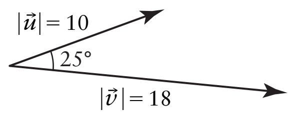
b) $\vec{r} = [-2, -8]$, $\vec{s} = [6, -1]$

c) $\vec{t} = [-7, 2]$, $\vec{u} = [6, 11]$

d) $\vec{e} = [2, 3]$, $\vec{f} = [9, -6]$

4. Determine the projection of $\vec{u}$ on $\vec{v}$.

a)



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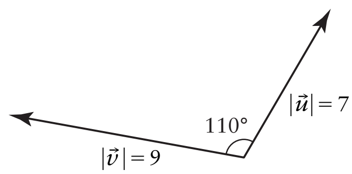
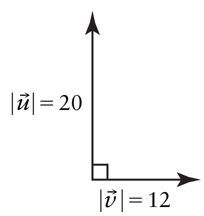


c)



5. In each case, determine the projection of the first vector on the second. Sketch each projection.

a) $\vec{a} = [6, -1]$, $\vec{b} = [11, 5]$

b) $\vec{c} = [2, 7]$, $\vec{d} = [-4, 3]$

c) $\vec{e} = [-2, -5]$, $\vec{f} = [-5, 1]$

d) $\vec{g} = [10, -3]$, $\vec{h} = [4, -4]$

B Connect and Apply

6. A factory worker pushes a package along a broken conveyor belt from $(-4, 0)$ to $(4, 0)$ with a 50-N force at a 30° angle to the conveyor belt. How much mechanical work is done if the units of the conveyor belt are in metres?

7. A force, $\vec{f}$, of 25 N is acting in the direction of $\vec{a} = [6, 1]$.

a) Find a unit vector in the direction of $\vec{a}$.

b) Find the Cartesian vector representing the force, $\vec{f}$, using your answer from part a).

c) The force $\vec{f}$ is exerted on an object moving from point $(4, 0)$ to point $(15, 0)$, with distance in metres. Determine the mechanical work done.

8. Justin applies a force at 20° to the horizontal to move a football tackling dummy 8 m horizontally. He does 150 J of mechanical work. What is the magnitude of the force?

9. Determine the angles of $\triangle$ABC for A(5, 1), B(4, −7), and C(−1, −8).

10. Consider the parallelogram with vertices at (0, 0), (3, 0), (5, 3), and (2, 3). Find the angles at which the diagonals of the parallelogram intersect.

11. The points P(−2, 1), Q(−6, 4), and R(4, 3) are three vertices of parallelogram PQRS.

a) Find the coordinates of S.

b) Find the measures of the interior angles of the parallelogram, to the nearest degree.

c) Find the measures of the angles between the diagonals of the parallelogram, to the nearest degree.

12. Determine the angle between vector $\overrightarrow{PQ}$ and the positive x-axis, given endpoints P(4, 7) and Q(8, 3).

13. Show that $|\text{proj}_{\vec{u}}\vec{v}| = |\vec{v}|\cos\theta$ can be written as $|\text{proj}_{\vec{u}}\vec{v}| = \dfrac{\vec{v}\cdot\vec{u}}{|\vec{u}|}$.

14. Show that $\text{proj}_{\vec{u}}\vec{v} = |\vec{v}|\cos\theta\left(\dfrac{1}{|\vec{u}|}\vec{u}\right)$ can be written as $\text{proj}_{\vec{u}}\vec{v} = \left(\dfrac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\right)\vec{u}$.

15. A store sells digital music players and DVD players. Suppose 42 digital music players are sold at \$115 each and 23 DVD players are sold at \$95 each. The vector $\vec{a} = [42, 23]$ can be called the sales vector and $\vec{b} = [115, 95]$ the price vector. Find $\vec{a}\cdot\vec{b}$ and interpret its meaning.

16. A car enters a curve on a highway. If the highway is banked 10° to the horizontal in the curve, show that the vector $[\cos 10°, \sin 10°]$ is parallel to the road surface. The x-axis is horizontal but perpendicular to the road lanes, and the y-axis is vertical. If the car has a mass of 1000 kg, find the component of the force of gravity along the road vector. The projection of the force of gravity on the road vector provides a force that helps the car turn. (Hint: The force of gravity is equal to the mass times the acceleration due to gravity.)

17. **Chapter Problem** The town of Oceanside lies at sea level and the town of Seaview is at an altitude of 84 m, at the end of a straight, smooth road that is 2.5 km long. Following an automobile accident, a tow truck is pulling a car up the road using a force, in newtons, defined by the vector $\vec{F} = [30\,000, 18\,000]$.

a) Find the force drawing the car up the hill and the force, perpendicular to the hill, tending to lift it.

b) What is the work done by the tow truck in pulling the car up the hill?

c) What is the work done in raising the altitude of the car?

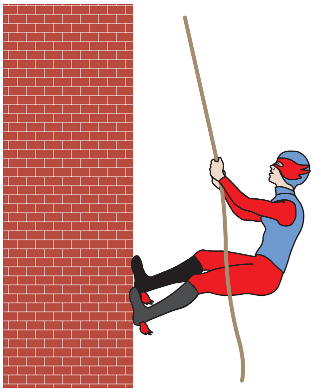
d) Explain the differences in your answers to parts b) and c).

18. How much work is done against gravity by a worker who carries a 25-kg carton up a 6-m-long set of stairs, inclined at 30°?

CONNECTIONS

A mass of 1 kg has a weight of about 9.8 N on Earth's surface.

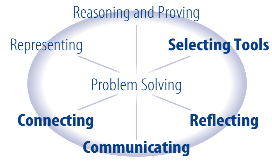
19. A superhero pulls herself 15 m up the side of a wall with a force of 234 N, at an angle of 12° to the vertical. What is the work done?



20. A crate is dragged 3 m along a smooth level floor by a 30-N force, applied at 25° to the floor. Then, it is pulled 4 m up a ramp inclined at 20° to the horizontal, using the same force. Then, the crate is dragged a further 5 m along a level platform using the same force again. Determine the total work done in moving the crate.

21. A square is defined by the unit vectors $\vec{i}$ and $\vec{j}$. Find the projections of $\vec{i}$ and $\vec{j}$ on each of the diagonals of the square.

22. The ramp to the loading dock at a car parts plant is inclined at 20° to the horizontal. A pallet of parts is moved 5 m up the ramp by a force of 5000 N, at an angle of 15° to the surface of the ramp.



a) What information do you need to calculate the work done in moving the pallet along the ramp?

b) Calculate the work done.

23. Draw a diagram to illustrate the meaning of each projection.

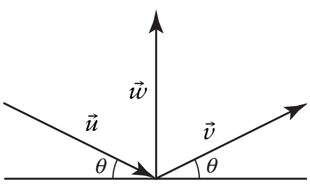
a) $\text{proj}_{\text{proj}_{\vec{v}}\,\vec{u}}\,\vec{u}$

b) $\text{proj}_{\vec{u}}\,(\text{proj}_{\text{proj}_{\vec{v}}}\,\vec{u})$

c) $\text{proj}_{\vec{v}}\,(\text{proj}_{\vec{v}}\,\vec{u})$

C Extend and Challenge

24. In light reflection, the angle of incidence is equal to the angle of reflection. Let $\vec{u}$ be the unit vector in the direction of incidence. Let $\vec{v}$ be the unit vector in the direction of reflection. Let $\vec{w}$ be the unit vector perpendicular to the face of the reflecting surface. Show that $\vec{v} = \vec{u} - 2(\vec{u} \cdot \vec{w})\vec{w}$.



25. a) Given vectors $\vec{a} = [6, 5]$ and $\vec{b} = [1, 3]$, find $\text{proj}_{\vec{b}}\,\vec{a}$.

b) Resolve $\vec{a}$ into perpendicular components, one of which is in the direction of $\vec{b}$.

26. Consider your answer to question 25. Resolve $\vec{F} = [25, 18]$ into perpendicular components, one of which is in the direction of $\vec{u} = [2, 5]$.

27. Describe when each of the following is true, and illustrate with an example.

a) $\text{proj}_{\vec{v}}\,\vec{u} = \text{proj}_{\vec{u}}\,\vec{v}$

b) $|\text{proj}_{\vec{v}}\,\vec{u}| = |\text{proj}_{\vec{u}}\,\vec{v}|$

28. **Math Contest** The side lengths of a right triangle are in the ratio 3 : 4 : 5. If the length of one of the three altitudes is 60 cm, what is the greatest possible area of this triangle?

29. **Math Contest** If $f(x) = \dfrac{1471}{n}\log_u x + \dfrac{538x}{u^n}$, where u and n are constants, determine $f(u^n)$.