INTEGRALES TRIGONOMÉTRICAS

Soluciones

1. $\int \frac{1}{3}\cdot \cos 7x \cdot \cos 2x \cdot dx = \left\{ \begin{array}{c} \text{Teniendo en cuenta :} \\ \cos mx \cdot \cos nx = \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right] \\ \cos 7x \cdot \cos 2x = \frac{1}{2}\left[\cos 5x + \cos 9x\right] \end{array} \right\} =$

$$= \int \frac{1}{3}\cdot\frac{1}{2}\left[\cos 5x + \cos 9x\right]\cdot dx = \frac{1}{6}\left[\int \cos 5x \cdot dx + \int \cos 9x \cdot dx\right] = \frac{1}{6}\left[\frac{1}{5}\,\text{sen}\,5x + \frac{1}{9}\,\text{sen}\,9x\right] + C =$$

$$= \frac{1}{30}\cdot \text{sen}\,5x + \frac{1}{54}\,\text{sen}\,9x + C$$

2. $\int (\cos x)^5 \cdot dx = \int \left(\cos^2 x\right)^2 \cdot \cos x \cdot dx = \int \left(1-\text{sen}^2 x\right)^2 \cdot \cos x \cdot dx = \left\{ \begin{array}{c} \text{sen}\,x = t \\ \cos x \cdot dx = dt \end{array} \right\} =$

$$\int \left(1-t^2\right)^2 \cdot dt = \int \left(1-2t^2+t^4\right)\cdot dt = t - 2\frac{t^3}{3} + \frac{t^5}{5} + C = \text{sen}\,x - \frac{2}{3}\text{sen}^3 x + \frac{1}{5}\text{sen}^5 x + C$$

3. $\int \text{sen}^2 x \cdot dx$ Teniendo en cuenta las relaciones trigonométricas:

$$\left. \begin{array}{r} \text{sen}^2 x + \cos^2 x = 1 \\ \cos^2 x - \text{sen}^2 x = \cos 2x \end{array} \right\} \Rightarrow \text{Restando : } 2\cdot\text{sen}^2 x = 1 - \cos 2x \quad \text{despejando} \quad \text{sen}^2 x = \frac{1}{2}\left(1-\cos 2x\right)$$

$$\int \text{sen}^2 x \cdot dx = \int \frac{1}{2}\left(1-\cos 2x\right)\cdot dx = \frac{1}{2}\left(x - \frac{1}{2}\text{sen}2x\right) + C$$

4. $\int \frac{\cos^5 x}{\text{sen}^2 x}\cdot dx = \int \frac{\cos^4 x \cdot \cos x}{\text{sen}^2 x}\cdot dx = \int \frac{\left(1-\text{sen}^2 x\right)^2 \cdot \cos x}{\text{sen}^2 x}\cdot dx = \left\{ \begin{array}{c} \text{sen}\,x = t \\ \cos x \cdot dx = t \end{array} \right\} = \int \frac{\left(1-t^2\right)^2}{t^2}\cdot dt =$

$$= \int \frac{1-2t^2+t^4}{t^2}\cdot dt = \int\left(\frac{1}{t^2} - 2 + t^2\right)\cdot dt = \int\left(t^{-2} - 2 + t^2\right)\cdot dt = \frac{t^{-1}}{-1} - 2t + \frac{t^3}{3} + C = \frac{-1}{t} - 2t + \frac{t^3}{3} + C =$$

$$= \frac{-1}{t} - 2t + \frac{t^3}{3} + C = \{\text{sen}\,x = t\} = \frac{\text{sen}^3 x}{3} - 2\text{sen}\,x - \frac{1}{\text{sen}\,x} + C$$

5. $\int \frac{1}{\text{sen}\,x - \text{tg}\,x}\cdot dx = \left\{ \begin{array}{c} \text{tg}\frac{x}{2} = t : dx = \frac{2dt}{1+t^2} \\ \text{sen}\,x = \frac{2t}{1+t^2} \\ \text{tg}\,x = \frac{2t}{1-t^2} \end{array} \right\} = \int \frac{1}{\frac{2t}{1+t^2} - \frac{2t}{1-t^2}}\cdot\frac{2dt}{1+t^2} = \int \frac{\left(1+t^2\right)\cdot\left(1-t^2\right)}{2t\left(1-t^2-\left(1+t^2\right)\right)}\cdot\frac{2dt}{1+t^2} =$

$$= \int \frac{1-t^2}{-2t^3}\cdot dt = \frac{1}{2}\int \frac{t^2-1}{t^3}\cdot dt = \frac{1}{2}\int\left(\frac{1}{t} - t^{-3}\right)\cdot dt = \frac{1}{2}\left(\text{Ln}\,t - \frac{t^{-2}}{-2}\right) + C = \frac{1}{2}\left(\text{Ln}\,t + \frac{1}{2t^2}\right) + C =$$

$$= \left\{ \begin{array}{l} \operatorname{tg}\frac{x}{2} = t \\ \operatorname{tg}\frac{x}{2} = \sqrt{\frac{1-\cos x}{1+\cos x}} \end{array} \right] : t = \sqrt{\frac{1-\cos x}{1+\cos x}} \right\} = \frac{1}{2}\left(\operatorname{Ln}\left|\sqrt{\frac{1-\cos x}{1+\cos x}}\right| + \frac{1}{2\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)^2} \right) + C =$$

$$= \frac{1}{4}\operatorname{Ln}\left|\frac{1-\cos x}{1+\cos x}\right| + \frac{1}{4}\cdot\frac{1+\cos x}{1-\cos x} + C$$

6. $\int \sqrt{\operatorname{sen} x}\cdot\cos^3 x\cdot dx = \int \sqrt{\operatorname{sen} x}\cdot\cos^2 x\cdot\cos x\cdot dx = \int \sqrt{\operatorname{sen} x}\cdot\left(1-\operatorname{sen}^2 x\right)\cdot\cos x\cdot dx = \left\{ \begin{array}{c} \operatorname{sen} x = t^2 \\ \cos x\cdot dx = 2t\cdot dt \end{array} \right\} =$

$$= \int \sqrt{t^2}\cdot\left(1-\left(t^2\right)^2\right)\cdot 2t\cdot dt = \left\{ \begin{array}{c} \operatorname{sen} x = t^2 \\ \cos x\cdot dx = 2t\cdot dt \end{array} \right\} = \int 2t^2\left(1-t^4\right)\cdot dt = 2\int\left(t^2-t^6\right)\cdot dt =$$

$$= 2\cdot\left(\frac{t^3}{3}-\frac{t^7}{7}\right) + C = \left\{ \begin{array}{l} \operatorname{sen} x = t^2 \\ t = \sqrt{\operatorname{sen} x} \end{array} \right\} = 2\cdot\left(\frac{\left(\sqrt{\operatorname{sen} x}\right)^3}{3} - \frac{\left(\sqrt{\operatorname{sen} x}\right)^7}{7}\right) + C =$$

$$= 2\cdot\left(\frac{\operatorname{sen} x\cdot\sqrt{\operatorname{sen} x}}{3} - \frac{\operatorname{sen}^3 x\cdot\sqrt{\operatorname{sen} x}}{7}\right) + C = \frac{2}{21}\sqrt{\operatorname{sen} x}\cdot\left(7\operatorname{sen} x - 3\operatorname{sen}^3 x\right) + C$$

7. $\int \frac{\operatorname{sen} x + \cos x}{\operatorname{sen} x - \cos x}\cdot dx = \int \frac{(\operatorname{sen} x+\cos x)\cdot(\operatorname{sen} x-\cos x)}{(\operatorname{sen} x-\cos x)\cdot(\operatorname{sen} x-\cos x)}\cdot dx = \int \frac{\operatorname{sen}^2 x - \cos^2 x}{(\operatorname{sen} x-\cos x)^2}\cdot dx =$

$$= \int \frac{-\left(\cos^2 x - \operatorname{sen}^2 x\right)}{\operatorname{sen}^2 x + \cos^2 x - 2\operatorname{sen} x\cdot\cos x}\cdot dx = \int \frac{-\cos 2x}{1-\operatorname{sen}2x}\cdot dx = \frac{1}{2}\int \frac{-\cos 2x\cdot(2)}{1-\operatorname{sen}2x}\cdot dx = \frac{1}{2}\operatorname{Ln}\left|1-\operatorname{sen}2x\right| + C$$

8. $\int \frac{dx}{\operatorname{sen} x} = \left\{ \begin{array}{l} \operatorname{tg}\frac{x}{2} = t : dx = \frac{2dt}{1+t^2} \\ \operatorname{sen} x = \frac{2t}{1+t^2} \end{array} \right\} = \int \frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2}} = \int \frac{dt}{t} = \operatorname{Ln}|t| + C = \left\{ t = \operatorname{tg}\frac{x}{2} = \sqrt{\frac{1-\cos x}{1+\cos x}} \right\} =$

$$= \operatorname{Ln}\left|\sqrt{\frac{1-\cos x}{1+\cos x}}\right| + C$$

9. $\int \frac{1}{1+\cos x}\cdot dx = \left\{ \begin{array}{l} \operatorname{tg}\frac{x}{2} = t : dx = \frac{2dt}{1+t^2} \\ \cos x = \frac{1-t^2}{1+t^2} \end{array} \right\} = \int \frac{1}{1+\frac{1-t^2}{1+t^2}}\cdot\frac{2dt}{1+t^2} = \int \frac{1+t^2}{1+t^2+1-t^2}\cdot\frac{2dt}{1+t^2} = \int dt = t + C =$

$$= \left\{ t = \operatorname{tg}\frac{x}{2} = \sqrt{\frac{1-\cos x}{1+\cos x}} \right\} = \sqrt{\frac{1-\cos x}{1+\cos x}} + C = \frac{\operatorname{sen} x}{1+\cos x} + C$$

10. $\int \frac{\operatorname{sen}^5 x}{\cos x}\cdot dx = \int \frac{\operatorname{sen}^4 x\cdot\operatorname{sen} x}{\cos x}\cdot dx = \int \frac{\left(1-\cos^2 x\right)^2\cdot\operatorname{sen} x}{\cos x}\cdot dx = \left\{ \begin{array}{c} \cos x = t \\ -\operatorname{sen} x\cdot dx = dt \end{array} \right\} = \int \frac{\left(1-t^2\right)^2}{t}\cdot(-dt) =$

$$= -\int \frac{1-2t^2+t^4}{t}\cdot dt = -\int\left(\frac{1}{t} - 2t + t^3\right)\cdot dt = -\left(\operatorname{Ln}|t| - \frac{2t^2}{2} + \frac{t^4}{4}\right) + C = \{t = \cos x\} =$$

$$= \cos^2 x - \frac{1}{4}\cos^4 x - \operatorname{Ln}|\cos x| + C$$

11. $\int \frac{sen^3 x}{2+\cos x}\cdot dx = \int \frac{sen^2 x}{2+\cos x} sen\, x\cdot dx = \int \frac{1-\cos^2 x}{2+\cos x} sen\, x\cdot dx = \left\{\begin{matrix} \cos x = t \\ -sen\, x\cdot dx = dt \end{matrix}\right\} = \int \frac{1-t^2}{2+t}(-dx) =$

$$= \int \frac{t^2-1}{2+t} dx = \left\{\begin{matrix} \text{Descomponiendo} \\ \text{por división} \\ \text{polinómica} \end{matrix}\right\} = \int \left(t-2+\frac{3}{2+t}\right) dx = \frac{t^2}{2} - 2t + 3Ln|2+t| + C = \{t=\cos x\} =$$

$$= \frac{\cos^2 x}{2} - 2\cos x + 3Ln|2+\cos x| + C$$

12. $\int \frac{sen\, 2x}{1+sen\, 2x}\cdot dx$ A partir de la expresión de la tangente del ángulo mitad, y teniendo en cuenta que si:

$$tg\frac{x}{2} = \sqrt{\frac{1-\cos x}{1+\cos x}} \Rightarrow tg\, x = \sqrt{\frac{1-\cos 2x}{1+\cos 2x}}$$

se puede despejar el coseno del ángulo doble en función de la tangente del ángulo

$$\cos 2x = \frac{1-tg^2 x}{1+tg^2 x}$$

Mediante la ecuación $sen^2\alpha + \cos^2\alpha = 1$ se despeja el seno del ángulo doble en función de la tangente del ángulo

$$sen\, 2x = \sqrt{1-\cos^2 2x} = \sqrt{1-\left(\frac{1-tg^2 x}{1+tg^2 x}\right)^2} = \frac{2tg\, x}{1+tg^2 x}$$

Haciendo el cambio de variable

$$tg\, x = t \to \left(tg^2 x + 1\right)\cdot dx = dt : dx = \frac{dt}{tg^2 x + 1} = \frac{dt}{t^2+1} : sen2x = \frac{2t}{t^2+1}$$

$$\int \frac{sen\, 2x}{1+sen\, 2x}\cdot dx = \int \frac{\frac{2t}{t^2+1}}{1+\frac{2t}{t^2+1}}\cdot\frac{dt}{t^2+1} = \int \frac{2t}{\left(t^2+2t+1\right)\cdot\left(t^2+1\right)}\cdot dt = \int \frac{2t}{(t+1)^2\cdot\left(t^2+1\right)}\cdot dt$$

Integral racional con una raíz real de multiplicidad dos y dos raíces imaginarias. La expresión se descompone en fracciones simples de las siguiente forma

$$\frac{2t}{(t+1)^2\cdot\left(t^2+1\right)} = \frac{A_1}{t+1} + \frac{A_2}{(t+1)^2} + \frac{M\cdot t+N}{t^2+1}$$

Las constantes se calculan sumando el segundo miembro e identificando los numeradores de las dos fracciones.

$$\frac{2t}{(t+1)^2\cdot\left(t^2+1\right)} = \frac{A_1(t+1)\left(t^2+1\right) + A_2\left(t^2+1\right) + (M\cdot t+N)\cdot(t+1)^2}{(t+1)^2\cdot\left(t^2+1\right)}$$

$$2t = A_1(t+1)\left(t^2+1\right) + A_2\left(t^2+1\right) + (M\cdot t+N)(t+1)^2$$

desarrollando y ordenando el segundo miembro de la igualdad

$$2t = (A_1+M)t^3 + (A_1+A_2+2M+N)t^2 + (A_1+M+2N)t + (A_1+A_2+N)$$

identificando por términos se plantea un sistema de cuatro ecuaciones con cuatro incógnitas

$$\left\{\begin{matrix} t^3 : A_1+M=0 \\ t^2 : A_1+A_2+2M+N=0 \\ t : A_1+M+2N=0 \\ Ind : A_1+A_2+N=0 \end{matrix}\right. \xrightarrow{\text{Resolviendo}} \left\{\begin{matrix} A_1=0 \\ A_2=-1 \\ M=0 \\ N=1 \end{matrix}\right.$$

$$\frac{2t}{(t+1)^2\cdot(t^2+1)}=\frac{-1}{(t+1)^2}+\frac{1}{t^2+1}$$

$$\int\frac{2t}{(t+1)^2\cdot(t^2+1)}dt=\int\left(\frac{-1}{(t+1)^2}+\frac{1}{t^2+1}\right)dt=\frac{1}{t+1}+\text{arctg}\, t+C=\{t=\text{tg}\, x\}=\frac{1}{\text{tg}\, x+1}+\text{arctg}(\text{tg}\, x)+C=$$

$$=\frac{1}{\text{tg}\, x+1}+x+C=\frac{\cos x}{\text{sen}\, x+\cos x}+x+C$$

13. $\displaystyle\int\left(\frac{-\sec x}{1+\text{tg}\, x}\right)^2\cdot dx=\int\frac{(-\sec x)^2}{(1+\text{tg}\, x)^2}\cdot dx=\int(1+\text{tg}\, x)^{-2}\cdot\sec^2 x\cdot dx=\int(1+\text{tg}\, x)^{-2}\cdot\frac{1}{\cos^2 x}\cdot dx=$

teniendo en cuenta la primitiva $\displaystyle\int f^n(x)\cdot f'(x)dx=\frac{f^{n+1}(x)}{n+1}+C\quad\forall n\neq-1$

$$=\frac{(1+\text{tg}\, x)^{-1}}{-1}+C=\frac{-1}{1+\text{tg}\, x}+C$$

14. $\int\cos^2 3x\cdot dx$ Teniendo en cuenta las relaciones trigonométricas:

$$\left.\begin{array}{r}\text{sen}^2 3x+\cos^2 3x=1\\ \cos^2 3x-\text{sen}^2 3x=\cos 6x\end{array}\right\}\Rightarrow \text{Restando}: 2\cdot\cos^2 3x=1+\cos 6x\quad \text{despejando}\quad \cos^2 x=\frac{1}{2}(1+\cos 6x)$$

$$\int\cos^2 3x\cdot dx=\int\frac{1}{2}(1+\cos 6x)\cdot dx=\frac{1}{2}\left(x+\frac{1}{6}\text{sen}\, 6x\right)+C$$

15. $\displaystyle\int\frac{\cos x}{4+3\text{sen}\, x}\cdot dx=\frac{1}{3}\int\frac{\cos x\cdot(3)}{4+3\text{sen}\, x}\cdot dx=\frac{1}{3}\text{Ln}(4+3\text{sen}\, x)+C$

16. $\displaystyle\int\cos x\cdot\text{ctg}^2 x\cdot dx=\int\cos x\cdot\frac{\cos^2 x}{\text{sen}^2 x}\cdot dx=\int\frac{(1-\text{sen}^2 x)}{\text{sen}^2 x}\cos x\cdot dx=\left\{\begin{array}{c}\text{sen}\, x=t\\ \cos x\cdot dx=dt\end{array}\right\}=\int\frac{1-t^2}{t^2}\cdot dt=$

$$=\int\left(\frac{1}{t^2}-1\right)\cdot dt=\int(t^{-2}-1)\cdot dt=\frac{t^{-1}}{-1}-t+C=\frac{-1}{t}-t+C=\{t=\text{sen}\, x\}=\frac{-1}{\text{sen}\, x}-\text{sen}\, x+C$$