Which measure for PFE?
The Risk Appetite Measure, ∆

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PFE, potential future exposure, is key risk control metric for trading desks
Credit officers set limits based on a single or multiple horizon PFE
Internal definition typically a quantile \((q)\) of positive mark-to-market (MtM) over a horizon \((T)\)

\[
PFE(q, T) := \max_{t \in [0, T]} \left[ \max_x \text{s.t. } P\{MtM^+ \geq x\} \geq q \right]
\]  

(1)

The key question is where does the distribution of MtM come from?
An introduction to PFE can be found in (Canabarro and Duffie 2003)
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2. Historical $P$ vs Risk-Neutral $Q$
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Historical ($\mathbb{P}$) vs Risk-Neutral ($\mathbb{Q}$) Calibrations

- **Historical Calibration**
  - Calibrate to a historical period
  - Within **limits** stakeholders have significant flexibility

- **Risk-Neutral Calibration**
  - Risk-neutral pricing provides prices at $t = 0$
    - Binary options on portfolio value give discounted-probability distribution
  - Inverse of market price of unit payment gives PFE
  - Measure-independent if market is complete

- **Mixed**
  - Risk-Neutral for market observables
  - Historical for non-observables, e.g. correlations
  - Often pick-and choose approach by stakeholders
Use Test in (BCBS-189 2011) suggests that risk factor evolution equations and portfolio values should pass historical backtesting

- Historical backtesting best practice described in (BCBS-185 2010)

FRTB-CVA (BCBS-325 2015) proposes two options for generating scenarios of discounted exposures used in CVA

- Option A: accounting-based CVA, i.e. whatever the Front Office is already doing, generally but not always market implied
- Option B: IMM-based CVA

Historical backtesting methods for counterparty credit risk (Anfuso, Karyampas, and Nawroth 2014; Kenyon and Stamm 2012) have lower power the more tests are done and the longer the horizon
Risk-Neutral Pricing

Theorem

**[First fundamental theorem of asset pricing]** If a market model has a risk-neutral probability measure, then it does not admit arbitrage.

Theorem

**[Second fundamental theorem of asset pricing]** A market model is complete (every security can be hedged) if and only if it has a risk-neutral probability measure that is unique.

- FACT TWO of (Brigo and Mercurio 2006) states

\[
\text{Price of Option}(K,T,S)_t = \mathbb{E}_t \left[ \frac{B(t)}{B(T)} \frac{\text{Payoff}(T)}{B(T)} \right]
\]

is invariant under change of numeraire from \( B \) to any other valid numeraire. Valid numeraires are non-zero self-financing portfolios.
Risk-Neutral Calibration

- Risk-neutral pricing provides prices at $t = 0$
- What is the value of GBP1 paid in 5Y, today?
  - Get market-implied price using riskless discount curve
- What is the probability of a 10Y-swap rate being above $K$ in 5Y?
  - Risk-neutral pricing will give the market-implied-discounted probability using a binary option on the 10Y-swap rate
  - Get market-implied probability by dividing by the value of GBP1 paid in 5Y
    - Discounting implicitly picks out the class of bank account and bond measures / numeraires
    - The implicit independence of probability and discounting further identifies the implicit risk-neutral measure as that with the bond numeraire, i.e. the T-Forward measure
      - (Stein 2013) previously made similar observations that additional criteria to fix the numeraire are required for risk-neutral probabilities to be well-defined.
- Market-implied independent discounting is unique but it is still market-implied, i.e. with price of risk $= 0$
Risk-Neutral = ?

- Full name for measures used in pricing:
  - Bank Account risk neutral measure (numeraire is continuously compounded bank account)
  - T-Forward risk neutral measure (numeraire is zero coupon bond maturity T)
  - T1-T2-Annuity risk neutral measure (numeraire is annuity from T1 to T2)

so what does “risk-neutral” mean?

- Risk-neutral means:
  - only expectation influences pricing
  - flat utility function: all outcomes have equal weight
  - the price of risk = 0
  - **not** mean-variance portfolio construction/pricing (i.e. local second-order utility function)
Difference can be described by the price of risk

**Price of Risk for Historical Calibration**

- In the usual Black-Scholes-Merton setup
  \[ m_M \equiv \frac{\mu - r}{\sigma} \]

  \( m_M \) is defined as the market price of risk (confusing terminology!)
- The **riskless** rate of return is \( r \)
- The rate of return on **open risk** is \( \mu \)
- The risk is the volatility of the return, i.e. \( \sigma \)

**Price of Risk for Risk-Neutral Calibration**

\[ m_{RN} \equiv 0 \]
How should open risk be priced? Standard answers:
- Assume there is no systematic risk and hence have price open risk at zero cost: market-implied pricing
- Mean-variance hedging
- Look at the real world and price accordingly

Limitations:
- Is there really no systematic risk?
- Does mean-variance align with institution's perception of risk?
- How do you calibrate a real world measure?
- For IMM banks risk factor dynamics must pass historical backtesting

Essentially we are discussing the price of risk
- Widely discussed in academic literature as the *market price of risk* (Berg 2010; Hull, Sokol, and White 2014)
- Will have a term structure (Hull, Sokol, and White 2014)
- Can have different prices for different risks (drift, volatility, correlation, etc.)
Market’s Price of Risk can be observed, in theory

In the usual Black-Scholes-Merton setup

$$m_r^M \equiv \frac{\mu - r}{\sigma}$$

$m_r^M$ is defined as the price of risk according to the market (M)

- The **riskless** rate of return is $r$
- The rate of return on **open risk** is $\mu$
- The risk is the volatility of the return, i.e. $\sigma$
Prices of Risk: $Q$-measure value

Price of risk

$0 = Q$-measure
Prices of Risk: $P$-measure calibration distribution, $t=0$
Prices of Risk: $\mathbb{P}$-measure calibration distribution, $t+1$. Calibration moves.

$0 = \mathbb{Q}$-measure

Price of risk

$0 = \mathbb{Q}$-measure

P-measure calibration likelihood distribution
Prices of Risk: regulatory-backtesting-consistent range, $t+2$. Calibration moves again.
Prices of Risk: regulatory-backtesting-consistent range, $t+3$. Range moves. How does a bank choose?

$0 = Q$-measure

Price of risk

Range consistent with regulatory backtesting at 95% confidence

$P$-measure
calibration
likelihood
distribution
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What is the Bank’s appetite for risk?

Definition

“Risk appetite is generally expressed through both quantitative and qualitative means and should consider extreme conditions, events, and outcomes. In addition, risk appetite should reflect potential impact on earnings, capital, and funding/liquidity.”

(Senior Supervisors Group, Observations on Developments in Risk Appetite Frameworks and IT Infrastructure, December 23, 2010)

- Observed in action — not a single utility function.
- No requirement for a coherent (Delbaen 2000) definition
- Part of Basel II with renewed emphasis post-crisis

Metrics and controls **already in place** describe the appetite that a bank has for risk. **Observable.**
Bank's price of risk can be observed defines the Risk Appetite Measure (RAM)

Bank Price of Interest Rate Risk

\[ m_B = \left( \frac{\text{Rates Desk Budget}}{\text{Rates Desk Investment}} - 1 \right) - r \]

- \( r \) is riskless rate
- \( \sigma_{\text{Rates VaR}} \) is the implied volatility from the Rates VaR limit

Economically profits can come from two sources
- rents, e.g. from monopoly position
- risk taking

Consider post-rent profits under normal competition
Prices of Risk: given a Bank’s Risk Appetite it has *already* chosen a price of risk

This defines **Risk Appetite Measure** (RAM). Complements $\mathbb{P}$, $\mathbb{Q}$ calibrations.
Assume that relative returns follow Normal distribution

- Fix desk budget rate of return, $\mu$
- Fix desk limit VaR($q$) in units of desk budget rate of return, $L$

\[
\sigma \Rightarrow = \frac{\mu (1 - L)}{\sqrt{2} \text{erfc}^{-1}(2q)}
\]

\[
m_B = \frac{\mu - r}{\sigma \Rightarrow}
\]

- Normal distribution has two parameters, so two constraints, desk budget and VaR limit, are sufficient
Implied Relative Return Volatility

Normal Relative Returns Model
VaR(95.), budget 10. percent

(c) C.Kenyon PFE using A
Risk Appetite Measure, A
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Implied Bank Price of Risk

From $\text{VaR}(\alpha)$, budget 10. percent

Normal Relative Returns Model

(c) C. Kenyon PFE using $A$ Risk Appetite Measure, $\Lambda$
Is price of risk robust under alternative return specification? Shifted-LogNormal relative returns

- LogNormal model is heavy-tailed and Shifted-LogNormal typical element of Forward models such as ZABR
- Fix shift at $-1$ (minus one hundred percent) assuming that at worst the desk can only lose all its investment
- Given budget and VaR limit have two constraints and two unknowns (given $\gamma = -1$), giving:

$$\sigma_{SLN} = - \sqrt{2} \text{erfc}^{-1}(2q) \pm \sqrt{2} \text{erfc}^{-1}(2q)^2 - 2(\log(L\mu - \gamma) - \log(\mu - \gamma))$$

$$\mu_{SLN} = \log(\mu - \gamma) - \frac{(\sigma_{SLN})^2}{2}$$

$$\sigma = \sqrt{\left(e^{\sigma_{SLN}^2} - 1\right) e^{2\mu_{SLN} + \sigma_{SLN}^2}}$$

$$m_B = \frac{\mu - r}{\sigma}$$
Implied Bank Price of Risk: Normal vs USLN

Price of Risk Implied from VaR Limit 0.95
Normal vs USLN Relative Returns Models

VaR limit (multiples of budget rate of return)

Implied Price of Risk
(excess relative return/SD)

- Normal
- Unit-Shifted-LogNormal
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PFE typically computed using historical or risk-neutral calibrations, i.e. by choosing a price of risk. However, historical approach has calibration issues and risk-neutral can fail.

- Historical backtesting, required by some regulations, provides limits on the price of risk. May not always include zero.

- Banks have observable risk appetite: defines **Risk Appetite Measure**
  - Uses bank price of risk
  - Bank risk appetite should be consistent with limits from historical backtesting

- Bank price of risk can be computed simply, and appears robust under change of assumed relative return distribution from Normal to Unit-Shifted-LogNormal
  - A zero price of risk is inconsistent with a finite risk appetite
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